Reconciling Number and Magnitude in Light of the Infinitesimal

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The beauty of mathematics stems at least in part from its eternal truth. A mathematical proposition true in Euclid's time will be just as true when Jesus gets around to calling his one thousand gross of souls home to Heaven for the Eternal Picnic. Yet, while it is always true, for instance, that the interior angles of a triangle equal two right angles, over the course of time men change their conceptions about certain aspects of mathematics, reformulating questions and tweaking axioms to account for new insights. It's not that mathematical truth ever goes out of style; it's merely that the metaconcepts that inform the structure of the axiomatic system itself come to be seen in a new light over time.

Aristotle's *Metaphysics*, from a certain point of view, limited the development of mathematics for almost 2000 years. To Aristotle, Ptolemy, and countless others, the concept of a finitely-sized sphere of the Universe placed an upper limit on mathematical magnitudes, and even on the quantities that could be comprehended by the finite human mind. We even see evidence of the blurring of physical and mathematical concepts in Aristotle's rejection of a mathematical void that stems from his proof of the non-existence of the physical void, which in turn depends on the concept of physical *place*.

Oresme, Pascal, Nicholas of Cusa, and Descartes, among others, took us a long way down the road of reformulating mathematical metaconcepts in an attempt to remove the limitations placed on mathematics by the long-entrenched desire of the ancients to prescribe universal philosophical principles that encompassed both the natural world and the ideal, and we can see the effects of these thinkers when Galileo boldly states, without defense, that a circle *is* an infinitely-sided polygon when he first lays out his polygon-rolling paradox. The concept of actualized infinity had so long seemed impossible that we almost shudder at such a claim. Descartes in particular irrevocably changed the fundamentals of mathematics by implicitly suggesting the existence of an underlying relationship between number, magnitude, and geometry; the ancients relegated these concepts to wholly disparate realms of understanding. Yet Apollonius was able to demonstrate with his limited tools an underlying relationship between the conic sections, a full explanation of which would be wanting until Descartes' analytic geometry. It seems that Galileo, in *Two New Sciences*, is stumbling upon a similar underlying relationship between number and magnitude which had been pragmatically glossed over by Descartes, but had not been fully explained. In dealing with finite concepts, this relationship can and did remain unenunciated, but Galileo's paradoxes point to the fact that a new formulation of the concepts of number and magnitude, which encompass and embrace each other in light of an actualized infinity, is necessary in order to address the infinitesimal without introducing paradox.

Had Galileo left well enough alone, he would have stood by Euclid's first two definitions. "A point is that which has no part," and "A line is breadthless length." These statements are to some extent self-explanatory, which is fitting for the nature of mathematical definitions. Yet according to these definitions, we should suspect that a point is of a fundamentally different kind than a line. This is the beginning of our inquiry.

A point, having no part, has no length, and thus is rightly called an indivisible. A line signifies length, and because one of these entities (the line) possesses a fundamental quality (length) that is lacking in the other (the point) by definition, we should deduce that they are fundamentally of a different kind. Further, in accordance with Aristotle's finitely-sized Universe, we are only guaranteed by Euclid to be able to produce a *finite* straight line continuously in a straight line (by the second postulate). Nowhere does Euclid even suggest that we can reverse this process, unproducing a finite straight line into the unfinite nondistance called a point.

Yet even in Euclid's geometry, points and lines are not wholly separate entities. We first note this in the third definition, where Euclid asserts that the extremities of a line are points. Thus any given line contains at least two points, and conversely (by the first postulate) two points determine a unique line. Further, we see later in Euclid's text that the intersection of two lines occurs in a point, and that the intersection of a line and a circle occurs either in one point or in two. From these statements, we see that lines contain points, or what is equivalent, points lie on lines. Granted, not all points lie on lines (e.g., the center of a circle), but any number of lines can *potentially* be drawn through the center.

Those who based their mathematics on Euclid, such as Pappus in his locus problem, never go so far as to claim that a line is *composed* of points. Pappus' proof allows us to find any finite number of points that fit the mathematical relationship under investigation, but in order to prove that the points so determined lie on a conic section, Pappus feels it necessary to divide the proof and to demonstrate, à la Apollonius, that *the line that contains these points* is the conic section. Galileo, on the other hand, would call the first part of Pappus' proof sufficient, claiming that the infinitely large *set* of points so determined *composes* the line; that is, that there is a one-to-one correspondence between the values that satisfy the ratio in question and the points on the line of the conic section.

The problem here is that to Pappus and all the Euclideans, the values that satisfy the ratio are *magnitudes*. But when the concept of incommensurability is suitably glossed over, à la Descartes, they can be understood as *numbers*. Numbers are of a fundamentally different kind than *points*. Points cannot be expressed in terms of ratio, so in order to solve with any degree of rigor the perplexing paradoxes on which Galileo stumbles, there is a need to explicitly address the relationship between numbers and their representation as points, and further, how the concept of a divisible magnitude is affected by such relationship.

In order to demonstrate that a line is in fact composed of points, we must understand the infinitesimal far better than we do when we merely give voice to phrases like "the limit of reduction of a finite magnitude," (although this is one way of obtaining an infinitesimal magnitude), or when we write "$\lim_{x\to 0}$." For in composing a line of points, Galileo is forced to speak of the number of points that compose a line, which, points having no part, must of course be infinitely many. This is the source of the polygon-rolling paradox, because then we must speak of every finite line as being composed of an infinite number of points. If lines represent magnitudes, then since all magnitudes are either less than, equal to, or greater than any other magnitude of the same kind, we are then forced to speak of an infinite number being likewise less than, equal to, or greater than another infinite number. Of course, this quickly regresses into semantic nonsense, which Galileo highlights in various ways in all three of the examples (rolling the infinitely-sided polygon, counting the squares and roots, and shrinking the soup dish) in the first day of *Two New Sciences*.

It seems that there are two possibilities for solving these paradoxes: either we must conclude that Galileo is treading on unjustified ground by so grossly intermingling number with geometry, or else we must further refine our concepts of number and magnitude to allow them to comingle, instead of relegating them to wholly disparate realms of understanding. The ancients, of course, would say that the former is the case. Yet in order that the advent of modern mathematical thought would not be a stillbirth, we should explore the latter option, pushing to the limit our conceptions of number and magnitude, and hopefully even revealing an underlying truth of which number and magnitude are both *representations*.

Let us at first explore Galileo's notion that a smaller circumference can measure a straight line equal in length to that measured by a larger circumference. Galileo's conclusion relies on the existence of an infinite number of infinitely small voids, which comes about from the example in which Galileo rotates the larger concentric circle and deduces the behavior of the smaller. There is another conclusion, though, stemming from the case of rotating the smaller circle and deducing the behavior of the larger. In this case, Galileo is forced to speak of an infinite number of infinitely small retrogressions. This is certainly a description of the process at hand, but it is not a suitable description, for two reasons.

The first reason is that he is forced to speak of all these so-called truths by analogy. That is to say, Galileo claims, as in the finitely-sided polygon, so there is something analogous (after making the mental leap into infinity) in the circle. We have seen countless theologians, making the same mental leap, flounder in trying to draw analogies between the incommensurable magnitudes of God and man, because such analogies leave us with a vague concept in mind of which we can only say, "If such an analogy exists between incommensurables, the consequent is of a fundamentally different kind than the antecedent." This sort of analogy is not a mathematical analogy, that is, a ratio, but a vague intuitive analogy that is invented in the imagination, rather than always having existed in the realm of the forms, or wherever it is that mathematics *exists*. For example, this writer has been fond lately of describing infinitesimal magnitudes as follows: "Zero is to the infinitely small as one is to the infinitely large." (The proof is left to the reader.) This "analogy" is somewhat descriptive, but caters only to the intuition and the imagination, and so this writer would never dream of submitting such an analogy to any even quasi-rigorous demonstration of the infinitesimal. Yet such is Galileo's analogy, and thus we should not accept it as sufficient.

The second reason may simply be an issue of mathematical elegance, although this writer claims that it points to imprecise thinking on Galileo's part: The two-fold explanation of the phenomenon at work in the case of rolling the infinitely-sided polygon is two-fold precisely because the model is viewed as a "phenomenon." In other words, Galileo is presenting a mechanical model of a mathematical construction. In understanding the model mechanically, we are permitted to speak of two different means (infinitely many voids on one hand, and infinitely many retrogressions on the other) to explain one mathematical relationship. Clearly, this is not acceptable to anyone who adheres to the necessity of rigorous mathematical demonstration.

Of course, Galileo's purpose is not a rigorous demonstration; it is more accurately a pedagogical endeavor, although his exposition of the paradoxes as such suggests that he did not have a rigorous demonstration at hand. Yet he may be handing us a potential tool in his digression on the problem of interleaved roots and squares. It is impossible to talk about this concept within any finite magnitude because the roots and squares paradox only manifests itself when we examine the entension to infinity of the patterns involved. Since the circumference of an infinitely-sided polygon approaches a finite ratio (π) to the diameter, we cannot rely on the manifestation on an infinite scale of the squares/roots problem for guidance, even for guidance by analogy.

In order to come to a greater understanding of the infinitesimal, consider the following two propositions:

Proposition 1 *Any finite magnitude can be divided into an infinite number of commensurable submagnitudes.*

Take any magnitude m, and let the sequence of natural numbers, $n = 1, 2, 3, \ldots$, be denominators. It follows that there exist an infinite number of commensurable submagnitudes a_n of lengths $a_1 = \frac{m}{1}$, $a_2 = \frac{m}{2}$, $a_3 = \frac{m}{3}$, and so on. That we can express the length of the submagnitudes as fractions demonstrates that all a_n are commensurable with m. Thus, any submagnitude a_n divides m a whole number of times (n times, to be exact).

A pattern emerges from this, and the result is perplexing.

Proposition 2 *Any infinitesimal magnitude can be divided into an infinite number of commensurable submagnitudes.*

Let the same m now be divided according to the sequence $b_n = \frac{m}{2n}$. The terms can be expressed as $b_1 = \frac{m}{2}$, $b_2 = \frac{m}{4}$, $b_3 = \frac{m}{6}$, and so on. Clearly, since n increases without bound, $2n$ increases without bound, and it follows that the magnitude m is also infinitely divisible (all divisions being commensurable with the whole) according to this rule.

Let m be divided again according to the sequence $c_n = \frac{m}{3n}$. The terms in this case can be expressed as $c_1 = \frac{m}{3}$, $c_2 = \frac{m}{6}$, $c_3 = \frac{m}{9}$, and so on. As above, $3m$ increases without bound. It follows from the existence of the sequences b_n and c_n that there exist at least two rules according to which m can be divided into an infinite number of submagnitudes, each of which is commensurable with the whole.

Taking the two rules together (that is, dividing the magnitude m simultaneously according to both rules) we find that we can again infinitely divide what is already infinitely divided. This is because we can always find a term b_n such that $c_{q-1} < b_n < c_q$ for any q. For example, $b_7 = \frac{m}{14}$ is smaller than $c_4 = \frac{m}{12}$ but larger than $c_5 = \frac{m}{15}$. Here we must retreat a little to the ancients' concept of arbitrarily small in order to discuss the result. Since there is some difficulty in talking about the "infiniteth" term of the sequence, let us say that as far out in the series b_n as we care to go, say to the 10^{145}th term, we will always find a term in c_n that subdivides the magnitude described by that b_n. This could be proved by mathematical induction.

Perhaps there is some need for clarification here. We are examining sequences of numbers, and by finding a part of m that has a ratio to it defined by any term in that sequence of numbers, we arrive at a submagnitude commensurable with the original magnitude. Since magnitude is infinitely divisible, we reach no limitations in our ability to further subdivide m, regardless of how much we have already divided it. A possible glitch in the method comes in being able to divide m infinitely, for there is just not enough time in the Universe to carry out all these pesky divisions. But let it have been done (to borrow Descartes' words), and see what falls out: Instead of allowing the sequence to determine the subdivisions, we can now state that for every commensurable subdivision there corresponds a term in an infinite sequence whose terms can be expressed as $\frac{m}{f(n)}$, for some $f(n)$. (In the case of b_n and c_n above, $f(n)$ is a linear function of n, but there is nothing to suggest that we are prohibited from dividing a magnitude according to a series determined by some whole-numbered exponent of n.) It is important to keep in mind here that we are not yet even discussing points. We are merely subdividing magnitudes according to number, something to which Descartes had no aversion.

Some clarification is also needed to explain why it is significant that the sequence of submagnitudes is cut by a second sequence. In examining a submagnitude defined by an arbitrarily large n value, without making the leap to actualized infinity, it is always possible to find a smaller submagnitude simply by looking at the next term in the sequence, $\frac{m}{f(n+1)}$. This possibility disappears after we let it have been done because we assume that the sequence is already divided into infinitesimals. The division according to the second sequence is significant because it is by this means that we can show that what we would call an infinitesimal is in fact further divisible. Witness the following:

If we continue the above division-according-to-sequence technique with, say, $d_n = \frac{m}{7n}$ and $e_n = \frac{m}{3n}$, examination reveals that all the magnitudes in d_n can be twice further divided. For example, $d_2 < e_4 < e_3 < d_1$. In fact, it would not be difficult to show that as the least common multiple of the coefficients in the denominators of any two sequences grows larger, more terms of one sequence are found between any two adjacent terms of the other sequence. Thus, as the least common multiple of the denominator coefficients of any two sequences approaches infinity, the number of terms of one sequence that lie between adjacent terms of the other sequence also approaches infinity. This suggests that, even with a finite but arbitrarily large LCM (we will not have let it be done quite yet), we can find an arbitrarily large number of terms of one sequence that lie between adjacent terms of the other. Now, since each term in the sequence represents an arbitrarily small subdivision of magnitude m, we have deduced that each arbitrarily small subdivision is itself divisible into an arbitrarily large number of commensurable parts.

Letting it have been done, we find something analogous (intuitively, not mathematically analogous) to the roots and squares paradox, but which takes place in a finite magnitude. In any infinitesimal submagnitude expressible as $\lim_{n\to\infty} d_n$, we can find an infinite number of subdivisions determined by $\lim_{n\to\infty} e_n$ as their least common multiple approaches infinity. Somehow, if this reasoning is valid, it suggests that there is no such thing as actualized infinity, because taking a magnitude posited to be infinitesimal, we find that it is itself infinitely divisible by something else. And there is nothing to make us doubt that that something else is again infinitely divisible. Aristotle would be relieved.

Now we should discuss the consequences of this conclusion when we leave the realm of magnitudes and begin to discuss this in terms of lines and points. A line is always cut at a point, and we discover, create, isolate, or whatever verb you choose to best visualize it, a point any and every time we cut a line. If, instead of creating submagnitudes, we envision that the result of the infinite sequence operation cuts a line of finite length m into lines in a given ratio of length to m, we must necessarily discover, or create, or what have you, points everywhere the line is cut.

Now the process of cutting the line creates infinitely many, infinitely small *lengths*, so we must, according to Euclid, find points at the ends of each infinitely short line segment. Yet the line segment, being infinitely small, it seems, should be a point, because if it were larger than a point, it could be further subdivided. So then we find ourselves imagining three points in a row, touching each other like the smallest conceivable circles, the middle point representing the infinitely small length, and the two endpoints being necessary to fulfill Euclid's definition of a line. Yet the line of reasoning about the infinite sequence operation concludes that we should always be able to find a point between two other points, because there will always be a magnitude that represents a length equal to a subdivision of the combined width of these three points, no matter how small. And, of course, the image of three discrete points touching is a blatantly mechanical model of the mathematics at hand, and thus should be regarded at best figuratively.

In fact, all of our visual reasoning about the infinitesimal seems in one way or another to reflect our predisposition to imagining these entities as physically extant, which is of course could be the beginning of a great misconception. It seems that the best course of action is to examine the properties of number and to forcibly reconstruct our intuitions about geometry to reflect what we discover about number, for the more prevalent in our mathematical thinking becomes the infinitesimal, the more it seems we need rigorous and demonstrable metaconcepts governing our conceptions of the particulars of the axiomatic system of geometry. The properties of number demonstrated by examination of convergent sequences should at least provide a suitable starting point.

There is, though, one snag to this course of action as well, and that is the problem of incommensurability. For despite a line being infinitely divisible according to an arithmetic sequence, we find that we must admit the line so composed is not continuous. There are indisputably voids which represent the lengths of fractional powers, for example, of n. For instance, despite that the line of length m is infinitely divisible an infinite number of times according to commensurable processes, we still have no point representing $\frac{m}{\sqrt{2}}$. Further, the more we think about incommensurable magnitudes, the more it seems likely that there are an infinite number of ways of infinitely dividing a magnitude according to sequences which yield magnitudes, and hence lengths, incommensurable with our original m. While these are not the particular voids to which Galileo makes reference in the polygon-rolling paradox, they nonetheless *are* voids, making every line horribly discontinuous. Regardless of our naturally mechanical intuitions about geometry, line is to some extent the definition of length, and our geometry starts to sound very atomistic if it admits a discrete, discontinuous definition of such a basic concept as length.

Now it turns out that the kind of infinity we discovered when we simultaneously divided the magnitude m according to a pair of sequences, as incomprehensibly large as it may seem, is still of a lower "degree" that that of the real numbers. Georg Cantor proved this in his 1891 paper, *On an Elemental Question in the Theory of Manifolds*. In essence, he shows that the rational numbers have a property called *countability*, which means that the entire set can be placed in sequential order. The real numbers, on the other hand, do not possess this property; their infinity is of a higher degree. Cantor deftly shows through a *reductio ad absurdum* proof that, assuming a complete and ordered list of the real numbers, we can always find at least one real number not on the list, which was originally supposed complete.

If we assume that the real numbers in a given interval share this property of uncountablility, which can be shown by a modification of Cantor's proof, we begin to see the problem underlying Galileo's polygon-rolling paradox. As mentioned at the outset of this paper, he assumes the existence of a one-to-one correspondence between the real numbers that satisfy the equation of a circle and the points on that circle. That the real numbers are not countable suggests that we should not immediately assume such a correspondence. In fact, even if we can rigorously prove a one-to-one correspondence between points and real numbers, we must admit that the uncountability of the real numbers precludes the adjacentness of the points to which they correspond. Thus Galileo's description of an infinite number of infinitesimal voids on the outer circle is not satisfying because the idea of the infinitesimal voids assumes some sort of natural adjacentness of the points that compose it. Cantor's uncountability proof demonstrates that Galileo is thinking in invalid terms by speaking of voids and retrogressions, and in fact offers the true but unsatisfying conclusion that we can not determine anything about the number of points on each of the two lines measured by the rolling of the two differently-sized circles. All we can say is that the resolution of the paradox cannot in any way assume adjacentness.

After Descartes we would be blind not to admit the striking similarities between number and geometry, but our inability to reconcile discoveries in number theory with geometry without paradox forces us to seek an underlying structure of which number and geometry are representations. In other words, since we cannot reconcile the offspring, we must look one generation back on the family tree.