

# Four out of Five Mathematicians Agree: Riemann is God

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Four out of five mathematicians also agree: The axioms of any formal mathematical system occupy a privileged role. They form the framework upon which the entire system hangs. In fact, any particular formal system can be said to be nothing but the set of logical consequences of its axioms. When the formal system corresponds to the regularity of sense experience, the axioms acquire greater significance in our minds, and we tend to ascribe what we call ‘reality’ or ‘existence’ to the principles upon which we have hung the system. Sciences are thus born. When the formal system bears little or no resemblance to sense experience, we call the result ‘pure’. But the means by which, say, Euclid or Russell and Whitehead, arrived at their axioms are surely carefully-guarded trade secrets. Maybe they are born in fits of inspiration. Maybe they are the bastard offspring of pencil shavings and cocktail napkins.

Aside from speculations on the mechanisms of genius, and regardless of the significance attributed to particular sets of axioms, they do unarguably occupy a privileged role. (One out of five mathematicians then is clearly just being contrary.) Axioms are meta-geometrical. They rest comfortably in their immunity from systematic considerations. In fact, they provide the touchstone by which alone the validity of systematic considerations is decided. But because they cannot be proved or disproved, axioms are the red-headed stepchildren of the logical world. Riemann agrees when he says, “the relation of these assumptions remains consequently in darkness; we neither perceive whether and how far their connection is necessary, nor, *a priori*, whether it is possible” (1).

*On the Hypotheses which Lie at the Bases of Geometry* is Riemann’s attempt to systematize the meta-geometrical. He knows better than to investigate the nature of mathematical systems by means of the interplay between inductive and deductive reasoning. In fact, he knows enough to leave alone all of the questions entertained above. He treats axioms, regardless of how they are realized, as what they are—*notions*—and thereby recasts geometry as the study of the relations among notions. By carefully examining the assumptions that lie behind geometrical notions, by removing their heretofore implicit dependence on sense experience, he takes the first steps necessary to enable the study of *possible* geometries. The geometry that we have come to know and love, first popularized in Euclid’s *Elements*, in the end retains its unique position merely because it happens to correspond to the stage upon which our sense experience is played out. Riemann’s success in deriving the properties of alternate spaces and manifolds of higher dimension is

made possible by the successful generalization of the notions that constitute geometrical relations.

But in order to see whence Riemann's suspicions arose, we must turn to one of his predecessors, Carl Friedrich Gauss, who studied the curvature of two-dimensional surfaces. The essential result of Gauss' *Disquisitiones Generales Circa Superficies Curvas* was to generalize the Euclidean notion of distance for curved surfaces, by way of a proof that the measure-relations on any surface are uniquely determined by its curvature. The first task Riemann sets for himself, "to discover the simplest matters of fact from which the measure-relations of space may be determined," is an extension of Gauss' work to manifolds of more than two dimensions.

The first notions to recast, in attempting to describe manifolds by means of measure-relations, are none other than those fundamental to measurement: magnitude-notions. Riemann characterizes them by saying, "magnitude-notions are only possible where there is an antecedent general notion which admits of different specialisations" (2). By so characterizing them, Riemann makes clear that magnitude-notions are, in the most general sense, organizing principles which make possible the quantification of relations among notions, that only when applied to geometrical considerations do they apply specifically to what we call measurement. His first hierarchical division distinguishes general notions that admit of continuous division from those that admit of discrete division. Only the former are geometrical in nature; common experience and set theory are characterized by the latter; the possibility of developing new discrete geometries is left to the reader.

The next hierarchical division along the branch of continuously-divided general notions distinguishes between manifolds that admit of measurement and those that do not. Only when magnitudes in a given manifold can be said to exist independently of position do they admit of measurement, that is, of meaningful "superposition of the magnitudes to be compared" (2). Otherwise, a common unit of measurement cannot be determined, and we must resign ourselves to greater-than and less-than relations. This branch of the hierarchy of possible relations among notions—continuous, measurable manifolds—are the focus of the remainder of Riemann's lecture. Having made the necessary *schnitts* in the science of extended magnitude, he confidently asserts that "nothing is assumed but what is contained in the notion of it." From this demarcation of the conceptual world, Riemann goes on to say that two further investigations "will make clear the true character of the an  $n$ -fold extent." The first of these involves the generalization of magnitude-notions in

order to construct multiply-extended magnitudes; the second involves the decomposition of multiply-extended magnitudes to reveal that “determinations of place” can be reduced to “determinations of quantity.”

Though Riemann’s language in the development of multiply-extended magnitudes seems deliberately vague—most notably in his condition that one must pass between specialisations of a continuous manifold “in a definite way” (whatever that means)—his formulation in fact maintains the most general character, which is in line with his goal not to introduce into the development of meta-geometrical notions assumptions that will color the outcome. Nevertheless, Riemann describes extended manifoldness in distinctly visual terms, revealing that, despite his careful use of the general language of ‘notions’, his thinking about these relations is distinctly geometrical. For instance, a simply extended manifoldness’ true character “is that in it a continuous progress from a point is possible only... forwards or backwards” (3). A doubly extended manifold is constructed when a singly extended one “passes over into another entirely different, and again in a definite way...” (3). No one would accuse Riemann of allowing these visual representations to affect his mathematical reasoning, but this is still a curious presentation of relationships between notions. His focus is clearly on the geometrical *representations* of the notions under investigation, which is valid because this branch of general notions has already been shown to correspond to geometry. And the specific visual descriptions—“pass over,” “forwards and backwards,” “in a definite way,”—are general enough in character not to prescribe a particular kind of geometry.

In the process of constructing the various degrees of multiply-extended magnitudes, Riemann stumbles upon the manifold corresponding to the physical space that sense experience indicates we inhabit. Space then turns out to be nothing but the “simplest case” (4) of a triply-extended manifold. But in reaching this conclusion, Riemann has somewhere made a transition from relations among notions to those that determine physical relations between objects. In the simplest analysis, this indicates Riemann follows Kant’s suspicions that space is a principle of the mind, or as Kant puts it, “nothing but the form of all appearances of outer sense. It is the subjective condition of sensibility...”<sup>1</sup> (71). But the manner in which Riemann repeatedly refers to space reveals that he attributes to it a kind of existence different from that of the other magnitude-notions constructed in the same fashion. Riemann, noting the agreement between the sense experience of

<sup>1</sup>I. Kant (1781), *Critique of Pure Reason*, translated by N. K. Smith (New York: Macmillan, 1929).

space and the formal system corresponding to it, ascribes real significance to the axioms that determine space, thus relegating all other manifolds to the realm of ‘pure’ mathematics. Curiously, Einstein required Riemann’s tools in order to develop a new notion of curved space in the general theory of relativity—a space that, due to our scope and scale, does not correspond to sense experience—but Riemann had no reason to suspect that his contribution to the science of extended magnitudes would have any impact on physics; he thought his work was pure.

The power to determine the measure-relations of a manifold depends on the second investigation mentioned above: its decomposition such that “the determination of position in an  $n$ -ply extended manifoldness is reduced to  $n$  determinations of quantity, and therefore the determination of position in a given manifoldness is reduced to a finite number of determinations of quantity. . .” (3). Given certain conditions corresponding to the notions of the geometrical entities under consideration—that the linear element, a function of the quantities  $dx$ , does not depend on the signs of those quantities; that the distance function must increase in all directions outward from a given point; that its first differential vanishes and its second derivative is positive—Riemann derives the characteristics that the distance function must possess, and thus also the form its expression takes in various manifolds: “consequently,  $ds$  is the square root of an always positive integral homogeneous function of the second order of the quantities  $dx$ , in which the coefficients are continuous functions of the quantities  $x$ ” (5). That is, a generalized linear element is given, in Einstein’s notation, by

$$ds = \sqrt{\sum_{\tau\sigma} g_{\sigma\tau} dx_{\sigma} dx_{\tau}}$$

where the terms in the  $g_{\sigma\tau}$  matrix are constants or functions of the quantities  $x$ . “Flat” manifolds are described by a matrix in which the coefficients of all terms for which  $\sigma \neq \tau$  are equal to zero.

It is important to note that Riemann’s quantities  $x$  and  $dx$  do not necessarily correspond to the coordinate axes as usually given. Earlier in the paper he asserts, “Position-fixing being reduced to quantity fixings, and the position of a point in the  $n$ -dimensioned manifoldness being consequently expressed by means of  $n$  variables  $x_1, x_2, x_3, \dots, x_n$ , the determination of a line comes to the giving of these quantities as functions of one variable” (4). Clearly Riemann has in mind a parametric representation of the variables, which need not correspond to an orthogonal coordinate system. He has already proved that any coordinate system adequate

to represent an  $n$ -dimensional manifold must necessarily include  $n$  functions of the parameter. Changing the coordinate system changes the  $n$  functions, but since the expression for the linear element contains  $n\frac{n+1}{2}$  unique coefficients,  $n\frac{n-1}{2}$  coefficients remain, “entirely determined by the nature of the continuum to be represented” (5). He goes on to say that “if, therefore, the curvature at each point in  $n\frac{n-1}{2}$  surface-directions is given, the measure-relations of the continuum may be determined from them. . .” (6). This is the crux of Riemann’s argument. The terms in the expression for the linear element that are not given by the coordinate system are given by the curvature, which implies, when discussing manifolds independently of the particular coordinate system chosen to represent them, that “the measure-relations of the manifoldness are entirely determined by the curvature” (8). Riemann has successfully generalized Gauss’ result.

We decided earlier that speculations on the mechanisms of genius are perhaps fruitless, but such speculations at least underscore the abrupt discontinuities in the progress of human knowledge. Riemann notes his debt to Gauss twice in this paper, and we can see that Gauss’ work may well have planted the suspicion in Riemann that his project could be undertaken with success. Yet nothing in his thinking required a world-view or empirical knowledge unavailable to pre-nineteenth century thinkers. Riemann unearthed never-before-conceived truths, broadening the science of geometry ten-fold, merely by carefully reexamining the notions that inform all of geometry, notions that countless geometers since Euclid had taken for granted. Whatever the nature of genius be, whatever the source of mathematical insight, Riemann clearly possessed both in unprecedented degrees. Hence it is no wonder why four out of five mathematicians agree: Riemann is God.

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